Effective Interactions of Relativisic Composite Particles in Unified Nonlinear Spinor-Field Models. II*

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Z. Naturforsch. **40 a,** 183–190 (1985); received September 15, 1984

Unified nonlinear spinor field models are selfregularizing quantum field theories in which all observable (elementary and non-elementary) particles are assumed to be bound states of fermionic preon fields. Due to their large masses the preons themselves are confined. In preceding papers a functional energy representation, the statistical interpretation and the dynamical equations were derived. In this paper the dynamics of composite particles is discussed. The composite particles are defined to be eigensolutions of the diagonal part of the energy representation. Corresponding calculations are in preparation, but in the present paper a suitable composite particle spectrum is assumed. It consists of preon-antipreon boson states and three-preon-fermion states with corresponding antifermions and contains bound states as well as preon scattering states. The state functional is expanded in terms of these composite particle states with inclusion of preon scattering states. The transformation of the functional energy representation of the spinor field into composite particle functional operators produces a hierarchy of effective interactions at the composite particle level, the leading terms of which are identical with the functional energy representation of a phenomenological boson-fermion coupling theory. This representation is valid as long as the processes are assumed to be below the energetic threshold for preon production or preon break-up reactions, respectively. From this it can be concluded that below the threshold the effective interactions of composite particles in a unified spinor field model lead to phenomenological coupling theories which depend in their properties on the bound state spectrum of the self-regularizing spinor theory.

PACS 11.10 Field theory
PACS 12.35 Composite Models of Particles

4. Cluster Functional Derivatives

In the preceding section we discussed the truncated functional equation (3.1) with respect to the definition of single cluster states. If the dynamics of cluster interactions is to be considered we need the complete functional equation (1.11) or (1.15), respectively, of our HDNSF model in the energy representation. According to our program we transform not only the states but also this equation into a cluster representation. In doing so the sources themselves as well as their derivatives have to be expressed in terms of cluster operators. For the sources the corresponding formulae are given by (3.14), (3.15a), (3.15b). We will express the derivatives in terms of cluster derivatives in this section by means of the chain rule. We assume that a functional state $|\mathfrak{F}\rangle$ of Eq. (1.15) is represented by (3.17). Then we have to describe the effect of one, two or three functional derivatives ∂_s, $\partial_s \partial_t$, $\partial_s \partial_t \partial_u$ etc. on $|\mathfrak{F}\rangle$ by means of cluster operators. In order to do this we substitute (3.8), (3.9a), (3.9b) into (3.17). This gives

$$
\begin{aligned}
|\mathfrak{F}\rangle = \sum_{NKL} & c(n_1 \dots n_N, q_1 \dots q_K, w_1 \dots w_L \,|\, a) \\
& \cdot C^{n_1 v_1}_{n_1} j_{u_1} \bar{j}_{v_1} \dots C^{u_N v_N}_{n_N} j_{u_N} \bar{j}_{v_N} \\
& \cdot C^{r_1 s_1 t_1}_{q_1} j_{r_1} j_{s_1} j_{t_1} \dots C^{r_k s_k t_k}_{q_k} j_{r_k} j_{s_k} j_{t_k} \\
& \cdot \bar{C}^{a_1 b_1 c_1}_{w_1} \bar{j}_{a_1} \bar{j}_{b_1} \bar{j}_{c_1} \dots \bar{C}^{a_L b_L c_L}_{w_L} \bar{j}_{a_L} \bar{j}_{b_L} \bar{j}_{c_L} \,|\, 0\rangle .
\end{aligned}
\tag{4.1}
$$

If we now observe the symmetry properties of $c(n_1 \dots n_N, q_1 \dots q_K, w_1 \dots w_L \,|\, a)$ as well as the antisymmetry of the cluster wave functions occurring in (4.1), then one can verify by a straight-forward calculation that

$$
\partial_s |\mathfrak{F}\rangle = \sum_n (\partial_s b_n) \frac{\delta}{\delta b_n} |\mathfrak{F}\rangle + \sum_q (\partial_s l_q) \frac{\delta}{\delta l_q} |\mathfrak{F}\rangle , \tag{4.2a}
$$

$$
\bar{\partial}_s |\mathfrak{F}\rangle = \sum_n (\bar{\partial}_s b_n) \frac{\delta}{\delta b_n} |\mathfrak{F}\rangle + \sum_w (\bar{\partial}_s \bar{l}_w) \frac{\delta}{\delta \bar{l}_w} |\mathfrak{F}\rangle , \tag{4.2b}
$$

i.e. the chain rule holds.

Along the same lines one can transform the second order derivatives, etc. For example, we still

* References see Part I, **40 a,** 14 (1985).

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write down a second order derivative explicitly

$$\begin{aligned}\partial_t\,\partial_s\,|\mathfrak{F}\rangle = &-\sum_{nn'}(\partial_s\,b_n)(\partial_t\,b_{n'})\frac{\delta}{\delta b_{n'}}\frac{\delta}{\delta b_n}|\mathfrak{F}\rangle\\ &-\sum_{nq'}(\partial_s\,b_n)(\partial_t\,l_{q'})\frac{\delta}{\delta l_{q'}}\frac{\delta}{\delta b_n}|\mathfrak{F}\rangle\\ &+\sum_{q}(\partial_t\,\partial_s\,l_q)\frac{\delta}{\delta l_q}|\mathfrak{F}\rangle\\ &+\sum_{qn}(\partial_s\,l_q)(\partial_t\,b_n)\frac{\delta}{\delta b_n}\frac{\delta}{\delta l_q}|\mathfrak{F}\rangle\\ &+\sum_{qq'}(\partial_s\,l_q)(\partial_t\,l_{q'})\frac{\delta}{\delta l_{q'}}\frac{\delta}{\delta l_q}|\mathfrak{F}\rangle. \qquad (4.3)\end{aligned}$$

As one can see from (4.2a), (4.2b) as well as from (4.3), there remain derivatives of cluster operators themselves. In the functional equation (1.15) these quantities can, however, be reexpressed in terms of cluster operators by means of the relations (3.14), (3.15a), (3.15b), i.e. a complete map of the functional elementary fermion derivatives into cluster operators is possible.

For brevity we demonstrate this possibility only for a single term of (1.15) which contains the second order derivative (4.3). In evaluating the cluster derivatives explicitly we obtain with (4.3)

$$\begin{aligned}&V_{\alpha\beta\gamma\delta}\,j_\alpha\,\bar{F}_{\gamma\gamma'}\,j_{\gamma'}\,\partial_\beta\,\partial_\delta\,|\mathfrak{F}\rangle\\ &=V_{\alpha\beta\gamma\delta}\,j_\alpha\,\bar{F}_{\gamma\gamma'}\,j_{\gamma'}\Bigg[-\sum_{nn'}C_n^{\delta v}\,\bar{j}_v\,C_{n'}^{\beta u}\,\bar{j}_u\,\frac{\delta}{\delta b_{n'}}\frac{\delta}{\delta b_n}\\ &\quad-\sum_{nq}C_n^{\delta v}\,\bar{j}_v\,3C_q^{\beta st}\,j_s\,j_t\,\frac{\delta}{\delta l_q}\frac{\delta}{\delta b_n}\\ &\quad+\sum_{q}6C_q^{\beta\delta v}\,j_u\frac{\delta}{\delta l_q} \qquad (4.4)\\ &\quad+\sum_{qn}3C_q^{\delta uv}\,j_u\,j_v\,C_n^{\beta s}\,\bar{j}_s\,\frac{\delta}{\delta b_n}\frac{\delta}{\delta l_q}\\ &\quad+\sum_{qq'}3C_q^{\delta uv}\,j_u\,j_v\,3C_{q'}^{\beta st}\,j_s\,j_t\,\frac{\delta}{\delta l_{q'}}\frac{\delta}{\delta l_q}\Bigg]|\mathfrak{F}\rangle\end{aligned}$$

and with (3.14), (3.15a), (3.15b) this gives

$$\begin{aligned}&V_{\alpha\beta\gamma\delta}\,j_\alpha\,\bar{F}_{\gamma\gamma'}\,j_{\gamma'}\,\partial_\beta\,\partial_\delta\,|\mathfrak{F}\rangle\\ &=V_{\alpha\beta\gamma\delta}\,\bar{F}_{\gamma\gamma'}\Bigg[-\sum_{\substack{nn'\\mm'}}C_n^{\delta v}\,C_{n'}^{\beta u}\,R_{\alpha v}^{m}\,R_{\gamma' u}^{m'}\,b_m\,b_{m'}\,\frac{\delta}{\delta b_{n'}}\frac{\delta}{\delta b_n}\\ &\quad+\sum_{\substack{nm\\qq'}}3C_q^{\delta uv}\,C_n^{\beta s}\,R_{\alpha\gamma' u}^{q'}\,R_{vs}^{m}\,l_{q'}\,b_m\,\frac{\delta}{\delta b_n}\frac{\delta}{\delta l_q}\\ &\quad+\sum_{\substack{qq'\\pp'}}3C_q^{\delta uv}\,3C_{q'}^{\beta st}\,R_{\alpha\gamma' u}^{p}\,R_{vst}^{p'}\,l_p\,l_{p'}\,\frac{\delta}{\delta l_{q'}}\frac{\delta}{\delta l_q}\\ &\quad-\sum_{\substack{mn\\qq'}}C_n^{\delta v}\,3C_q^{\beta st}\,R_{\alpha v}^{m}\,R_{\gamma' st}^{q'}\,b_m\,l_{q'}\,\frac{\delta}{\delta l_q}\frac{\delta}{\delta b_n}\\ &\quad+\sum_{qq'}6C_q^{\beta\delta u}\,R_{\alpha\gamma' u}^{q'}\,l_{q'}\,\frac{\delta}{\delta l_q}\Bigg]|\mathfrak{F}\rangle, \qquad (4.5)\end{aligned}$$

i.e. this term is completely transformed into the cluster representation. In the same way all other terms of (1.15) can be treated and it is interesting to note that this can be done unambiguously apart from certain terms where the special choice of the transformation will be discussed later on.

Concluding this section we may add: If elementary fermions would be allowed to also appear in the functional state $|\mathfrak{F}\rangle$, then the chain rule (4.2a), (4.2b) etc. would explicitly depend on the various numbers of the elementary fermions contained in the basis states of the expansion (4.1) as can easily be verified by direct calculation. In this case it would not be possible to find a state independent cluster representation of ∂_s and $\bar{\partial}_s$ etc. This means: the possibility of a simultaneous cluster representation of functional states as well as of the functional equation depends on the fulfillment of the threshold condition for the non-appearance of elementary fermions and is consistent with this condition.

5. Classification of Cluster Interactions

The transformation of the basic equation (1.15) from the elementary fermion representation into the bound state cluster representation is accompanied by a ramification of the original unified fermion interaction into the various cluster interactions. Before further analysis of the system is made, it is convenient to collect these various interactions in a systematic way. According to Assumption 1 we have to expect bb, ll, $\bar{l}\bar{l}$, $l\bar{l}$, bl, $b\bar{l}$ and $bl\bar{l}$ interactions. Therefore, after the transformation the operator $\mathscr{H}$ of (1.15) can be decomposed into

$$\begin{aligned}\mathscr{H}[j,\bar{j},\partial,\bar{\partial}] &= \mathscr{H}\left[b,l,\bar{l},\frac{\delta}{\delta b},\frac{\delta}{\delta l},\frac{\delta}{\delta\bar{l}}\right] \qquad (5.1)\\ &= \mathscr{H}_{bb}+\mathscr{H}_{ll}+\mathscr{H}_{\bar{l}\bar{l}}+\mathscr{H}_{l\bar{l}}+\mathscr{H}_{bl}+\mathscr{H}_{b\bar{l}}+\mathscr{H}_{bl\bar{l}},\end{aligned}$$

where the various suboperators follow directly from a rearrangement of the transformed terms of $\mathscr{H}$. Before giving the explicit form of these suboperators we consider the cluster eigenvalue equations themselves. As can be seen by comparison of (3.1) and (1.11) (and of course by definition) the truncated Eq. (3.1) is part of (1.11) and thus of (1.15), too.

In the abbreviated notation of (1.15) this part is given by

$$\begin{aligned}\mathscr{H}_{cl}[j,\bar{j},\partial,\bar{\partial}] &:= j_\alpha K_{\alpha\beta}\partial_\beta + \bar{j}_\alpha \bar{K}_{\alpha\beta}\bar{\partial}_\beta \qquad (5.2)\\ &+ V_{\alpha\beta\gamma\delta} j_\alpha [\bar{F}_{\gamma\gamma'} j_{\gamma'}\partial_\beta\partial_\delta - F_{\delta\delta'}\bar{j}_{\delta'}\partial_\beta\bar{\partial}_\gamma - F_{\beta\beta'}\bar{j}_{\beta'}\bar{\partial}_\gamma\partial_\delta]\\ &+ V_{\alpha\beta\gamma\delta}\bar{j}_\alpha [F_{\gamma\gamma'}\bar{j}_{\gamma'}\bar{\partial}_\beta\bar{\partial}_\delta - \bar{F}_{\delta\delta'} j_{\delta'}\bar{\partial}_\beta\partial_\gamma - \bar{F}_{\beta\beta'} j_{\beta'}\partial_\gamma\bar{\partial}_\delta]\end{aligned}$$

and the eigenvalue equation for the boson cluster $C_n^{\alpha\beta}$ can be obtained from (5.2) by projection. It reads

$$\begin{aligned}K_{\alpha\beta'} C_n^{\beta\beta} &+ \bar{K}_{\alpha\beta'} C_n^{\beta\beta'} - V_{\alpha\beta'\gamma\delta} F_{\beta'\beta}\, C_n^{\delta\gamma}\\ &+ V_{\alpha\delta'\gamma\delta} F_{\delta\beta} C_n^{\delta'\gamma} - V_{\alpha\beta'\gamma\delta}\bar{F}_{\beta\beta'} C_n^{\gamma\delta}\\ &+ V_{\alpha\delta'\gamma\delta}\bar{F}_{\delta\beta} C_n^{\gamma\delta'} = E_n^b C_n^{\alpha\beta}. \qquad (5.3)\end{aligned}$$

Similar equations can be derived for the fermion clusters by projection from (5.2). It should, however, be emphasized that the operator (5.2) does not only describe the single cluster states but also interactions between several clusters. But this is of no meaning with respect to the definition of single cluster states, and as (5.2) is part of (5.1) all many-cluster interaction terms of (5.2) are automatically contained in (5.1). The eigenvalue equation (5.3), however, can be used to simplify some terms of $\mathscr{H}_{bb}$, $\mathscr{H}_{ll}$ and $\mathscr{H}_{\bar{l}\bar{l}}$ in (5.1). In particular, the corresponding terms in $\mathscr{H}_{bb}$ are given by the expressions

$$\begin{aligned}\mathscr{H}_{bb}^0 |\mathfrak{F}\rangle &:= \sum_{n\,m}[K_{\alpha\beta'} C_n^{\beta'\beta} R_{\alpha\beta}^m + \bar{K}_{\alpha\beta'} C_n^{\beta\beta'} R_{\beta\alpha}^m\\ &- V_{\alpha\beta\gamma\delta} F_{\beta\beta'} C_n^{\delta\gamma} R_{\alpha\beta'}^m + V_{\alpha\beta\gamma\delta} F_{\delta\delta'} C_n^{\beta\gamma} R_{\alpha\delta'}^m\\ &- V_{\alpha\beta\gamma\delta}\bar{F}_{\beta\beta'} C_n^{\gamma\delta} R_{\beta'\alpha}^m + V_{\alpha\beta\gamma\delta}\bar{F}_{\delta\delta'} C_n^{\gamma\beta} R_{\delta'\alpha}^m]\\ &\cdot b_m \frac{\delta}{\delta b_n}|\mathfrak{F}\rangle. \qquad (5.4)\end{aligned}$$

After some relabeling the application of (5.3) then yields

$$\begin{aligned}\mathscr{H}_{bb}^0 |\mathfrak{F}\rangle &= \sum_{n\,m} E_n^b C_n^{\alpha\beta} R_{\alpha\beta}^m b_m \frac{\delta}{\delta b_n}|\mathfrak{F}\rangle\\ &= \sum_n E_n^b b_n \frac{\delta}{\delta b_n}|\mathfrak{F}\rangle \qquad (5.5)\end{aligned}$$

if (3.10) is taken into account.

Analogous expressions can be derived for the fermion and antifermion self-energy expressions $\mathscr{H}_{ll}^0$ and $\mathscr{H}_{\bar{l}\bar{l}}^0$ which for brevity we will not explicitly discuss here.

Using these simplifications the straight-forward transformation of (1.15) into the cluster representation gives the following expressions for the suboperators

$$\begin{aligned}\mathscr{H}_{bb} &= \sum_{k=0}^{5}\mathscr{H}_{bb}^k; \quad \mathscr{H}_{ll} = \sum_{k=0}^{1}\mathscr{H}_{ll}^k;\\ \mathscr{H}_{\bar{l}\bar{l}} &= \sum_{k=0}^{1}\mathscr{H}_{\bar{l}\bar{l}}^k; \quad \mathscr{H}_{bl} = \sum_{k=1}^{5}\mathscr{H}_{bl}^k;\\ \mathscr{H}_{b\bar{l}} &= \sum_{k=1}^{5}\mathscr{H}_{b\bar{l}}^k; \quad \mathscr{H}_{bl\bar{l}} = \sum_{k=1}^{3}\mathscr{H}_{bl\bar{l}}^k \qquad (5.6)\end{aligned}$$

with

$$\mathscr{H}_{bb}^0 := \sum_n E_n^b b_n \frac{\delta}{\delta b_n}; \qquad (5.7)$$

$$\begin{aligned}\mathscr{H}_{bb}^1 := &- \sum_{\substack{n\,n'\\ m\,m'}} V_{\alpha\beta\gamma\delta}\bar{F}_{\gamma\gamma'} C_n^{\delta v} C_{n'}^{\beta u} R_{\alpha v}^m R_{\gamma' u}^{m'}\\ &\cdot b_m b_{m'} \frac{\delta}{\delta b_{n'}}\frac{\delta}{\delta b_n} + 5 \text{ similar terms}; \qquad (5.8)\end{aligned}$$

$$\begin{aligned}\mathscr{H}_{bb}^2 := &- \sum_{\substack{n\,n'\,n''\\ m\,m'}} V_{\alpha\beta\gamma\delta} C_n^{\delta v} C_{n'}^{u\gamma} C_{n''}^{\beta z} R_{\alpha v}^m R_{uz}^{m'}\\ &\cdot b_m b_{m'} \frac{\delta}{\delta b_n}\frac{\delta}{\delta b_{n'}}\frac{\delta}{\delta b_{n''}} + 1 \text{ similar term}; \qquad (5.9)\end{aligned}$$

$$\begin{aligned}\mathscr{H}_{bb}^3 := &\sum_{n\,n'\,n''} V_{\alpha\beta\gamma\delta} C_n^{\delta\gamma} C_{n'}^{\beta v} R_{\alpha v}^{n''} b_{n''}\frac{\delta}{\delta b_{n'}}\frac{\delta}{\delta b_n}\\ &+ 3 \text{ similar terms}; \qquad (5.10)\end{aligned}$$

$$\begin{aligned}\mathscr{H}_{bb}^4 := &\sum_{n\,m\,m'} V_{\alpha\beta\gamma\delta}\bar{F}_{\beta\beta'} F_{\gamma\gamma'} C_n^{v\delta} R_{\beta'\alpha}^{m'} R_{v\gamma'}^{m'} b_m b_{m'}\frac{\delta}{\delta b_n}\\ &+ 5 \text{ similar terms}; \qquad (5.11)\end{aligned}$$

$$\begin{aligned}\mathscr{H}_{bb}^5 := &\sum_{n\,n'} V_{\alpha\beta\gamma\delta} F_{\beta\beta'}\bar{F}_{\gamma\gamma'} F_{\delta\delta'} R_{\alpha\beta'}^n R_{\gamma'\delta'}^{n'} b_{n'} b_n\\ &+ 1 \text{ similar term}; \qquad (5.12)\end{aligned}$$

$$\mathscr{H}_{ll}^0 := \sum_q E_q^l l_q \frac{\delta}{\delta l_q}; \qquad (5.13)$$

$$\begin{aligned}\mathscr{H}_{ll}^1 := &\sum_{\substack{q\,q'\\ p\,p'}} V_{\alpha\beta\gamma\delta}\bar{F}_{\gamma\gamma'} 3C_q^{\delta u v} 3C_{q'}^{\beta s t} R_{\alpha\gamma' u}^p R_{vst}^{p'}\\ &\cdot l_p l_{p'}\frac{\delta}{\delta l_{q'}}\frac{\delta}{\delta l_q}. \qquad (5.14)\end{aligned}$$

The expressions $\mathscr{H}^0_{ll}$ and $\mathscr{H}^1_{ll}$ are obtained from $\mathscr{H}^0_{l\bar{l}}$ and $\mathscr{H}^1_{l\bar{l}}$ by overlining the fermion quantities.

$$\mathscr{H}^1_{bl} := \sum_{nqq'} V_{\alpha\beta\gamma\delta}\, C_n^{\delta\gamma}\, 3C_q^{\beta u v}\, R^{q'}_{\alpha u v}\, l_{q'} \frac{\delta}{\delta l_q} \frac{\delta}{\delta b_n} + 1 \text{ similar term}; \tag{5.15}$$

$$+ \sum_{nqq'} V_{\alpha\beta\gamma\delta}\, 6C_q^{\beta\delta u}\, C_n^{v\gamma}\, R^{q'}_{\alpha u v}\, l_{q'} \frac{\delta}{\delta l_q} \frac{\delta}{\delta b_n};$$

$$\mathscr{H}^2_{bl} := \sum_{mqq'} V_{\alpha\beta\gamma\delta}\, \bar{F}_{\gamma\gamma'} F_{\delta\delta'}\, 3C_q^{\beta st}\, R^m_{\alpha\delta'}\, R^{q'}_{\gamma' st}\, b_m\, l_{q'} \frac{\delta}{\delta l_q} + 2 \text{ similar terms}; \tag{5.16}$$

$$\mathscr{H}^3_{bl} := -\sum_{nmqq'} V_{\alpha\beta\gamma\delta}\, \bar{F}_{\gamma\gamma'}\, C_n^{\delta v}\, 3C_q^{\beta st}\, R^m_{\alpha v}\, R^{q'}_{\gamma' st} \cdot b_m\, l_{q'} \frac{\delta}{\delta l_q} \frac{\delta}{\delta b_n} + 5 \text{ similar terms}; \tag{5.17}$$

$$\mathscr{H}^4_{bl} := \sum_{\substack{nn'm \\ qq'}} V_{\alpha\beta\gamma\delta}\, 3C_q^{\delta u v}\, C_n^{z\gamma}\, C_{n'}^{\beta\alpha}\, R^{q'}_{\alpha u v}\, R^m_{za} \cdot l_{q'}\, b_m \frac{\delta}{\delta b_{n'}} \frac{\delta}{\delta b_n} \frac{\delta}{\delta l_q} + 1 \text{ similar term}; \tag{5.18}$$

$$\mathscr{H}^5_{bl} := -\sum_{\substack{nqq' \\ pp'}} V_{\alpha\beta\gamma\delta}\, 3C_q^{\delta u v}\, C_n^{z\gamma}\, 3C_{q'}^{\beta ab}\, R^p_{\alpha u v}\, R^{p'}_{zab} \cdot l_p\, l_{p'} \frac{\delta}{\delta l_{q'}} \frac{\delta}{\delta l_q} \frac{\delta}{\delta b_n}. \tag{5.19}$$

The expressions $\mathscr{H}^1_{b\bar{l}} \ldots \mathscr{H}^5_{b\bar{l}}$ are obtained from $\mathscr{H}^1_{bl} \ldots \mathscr{H}^5_{bl}$ by overlining the fermion quantities.

$$\mathscr{H}^1_{bl\bar{l}} := \sum_{\substack{mm' \\ qw}} V_{\alpha\beta\gamma\delta}\, 6C_q^{\beta\delta v}\, 3C_w^{\gamma us}\, R^m_{\alpha u}\, R^{m'}_{vs} \cdot b_m\, b_{m'} \frac{\delta}{\delta \bar{l}_w} \frac{\delta}{\delta l_q} + 1 \text{ similar term}; \tag{5.20}$$

$$\mathscr{H}^2_{bl\bar{l}} := \sum_{\substack{nqq' \\ ww'}} V_{\alpha\beta\gamma\delta}\, C_n^{\delta v}\, 3\bar{C}_w^{\gamma ab}\, 3C_q^{\beta uz}\, R^{q'}_{\alpha uz}\, \bar{R}^{w'}_{vab} \cdot l_{q'}\, \bar{l}_{w'} \frac{\delta}{\delta l_q} \frac{\delta}{\delta \bar{l}_w} \frac{\delta}{\delta b_n} + 3 \text{ similar terms}; \tag{5.21}$$

$$\mathscr{H}^3_{bl\bar{l}} := \sum_{\substack{mm' \\ qq'q''w}} V_{\alpha\beta\gamma\delta}\, 3C_q^{\delta u v}\, 3\bar{C}_w^{\gamma ab}\, 3C_{q'}^{\beta st} \cdot R^{q''}_{\alpha u v}\, R^m_{as}\, R^{m'}_{bt}\, l_{q''}\, b_m\, b_{m'} \frac{\delta}{\delta l_{q'}} \frac{\delta}{\delta \bar{l}_w} \frac{\delta}{\delta l_q} + 1 \text{ similar term}; \tag{5.22}$$

$$\mathscr{H}_{l\bar{l}} := -\sum_{\substack{qq' \\ ww'}} V_{\alpha\beta\gamma\delta}\, F_{\beta\beta'}\, 3C_q^{\delta st}\, 3\bar{C}_w^{\gamma u v}\, R^{q'}_{\alpha st}\, \bar{R}^{w'}_{\beta' u v} \cdot l_{q'}\, \bar{l}_{w'} \frac{\delta}{\delta \bar{l}_w} \frac{\delta}{\delta l_q} + 3 \text{ similar terms}. \tag{5.23}$$

These terms will be evaluated in detail in the following sections.

6. Two-Fermion Cluster Wave Functions

The two-fermion sector of (3.1) was studied in [11]. Although the analytic expressions for the spinorial wave functions of the two-fermion clusters were not explicitly given in this paper, some general properties of the corresponding eigenvalue equations were derived which are useful for the determination of the wave functions themselves and which we will apply here.

The cluster wave functions $C_n^{u v}$ of Eq. (3.8) read in the full notation

$$C_n^{u v} \equiv \varphi_v^{(\pm)} \left(\begin{matrix} r & r' \\ \boldsymbol{r}, & \boldsymbol{r}' \\ \alpha & \beta \end{matrix} \middle| \boldsymbol{k}\right) = e^{\pm i\boldsymbol{k}(\boldsymbol{r}+\boldsymbol{r}')/2}\, \chi_v^{(\pm)} \left(\begin{matrix} r & r' \\ \boldsymbol{r} - & \boldsymbol{r}' \\ \alpha & \beta \end{matrix} \middle| \boldsymbol{k}\right) \tag{6.1}$$

where r, r' are the auxiliary field indices, α, β the spinorial indices, $(\pm)$ the positive and negative class indices, and v the internal cluster state quantum number. The equations for the determination of such states are formally given by (5.3). For practical calculations as well as for theoretical investigations it is, however, convenient to perform an orthogonal transformation of the sources with respect to the auxiliary fields. In the following only these auxiliary field indices are used, all other indices are omitted for brevity. The transformation reads [11]

$$j^1 = (2)^{-1/2}(j^\varrho + j^\chi); \quad j^\varrho = (2)^{-1/2}(j^1 + j^2); \tag{6.2}$$

$$j^2 = (2)^{-1/2}(j^\varrho - j^\chi); \quad j^\chi = (2)^{-1/2}(j^1 - j^2)$$

and induces the following transformation for (6.1)

$$\begin{pmatrix} \varphi^{11} \\ \varphi^{21} \\ \varphi^{12} \\ \varphi^{22} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{\varphi}^{\varrho\varrho} \\ \hat{\varphi}^{\chi\varrho} \\ \hat{\varphi}^{\varrho\chi} \\ \hat{\varphi}^{\chi\chi} \end{pmatrix}. \tag{6.3}$$

The norm of a cluster state $|a\rangle$ can be expressed in terms of left-hand and right-hand solutions σ^{ik} and φ^{ik} of Eqs. (5.3) and is given by [11]

$$\langle a|a\rangle = \sum_{ik} \sigma^{ik}\, \varphi^{ik}. \tag{6.4}$$

This expression must be invariant under orthogonal transformations of the sources. As the transformation matrix of (6.3) is also orthogonal, from this

invariance it follows that

$$\begin{pmatrix}\sigma^{11}\\ \sigma^{21}\\ \sigma^{12}\\ \sigma^{22}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}\hat{\sigma}^{\varrho\varrho}\\ \hat{\sigma}^{\chi\varrho}\\ \hat{\sigma}^{\varrho\chi}\\ \hat{\sigma}^{\chi\chi}\end{pmatrix}. \tag{6.5}$$

For the further evaluation it was shown in [11] that symmetry combinations must be introduced in order to obtain total antisymmetric wave functions. The transition to symmetry combinations can be achieved by means of a further orthogonal transformation

$$a=(2)^{-1/2}(\varphi^{\chi\varrho}+\varphi^{\varrho\chi});\quad \varphi^{\xi\varrho}=(2)^{-1/2}(a+s);$$

$$s=(2)^{-1/2}(\varphi^{\chi\varrho}-\varphi^{\varrho\chi});\quad \varphi^{\varrho\chi}=(2)^{-1/2}(a-s) \tag{6.6}$$

and

$$\alpha=(2)^{-1/2}(\sigma^{\chi\varrho}+\sigma^{\varrho\chi});\quad \sigma^{\chi\varrho}=(2)^{-1/2}(\alpha+\sigma);$$

$$\sigma=(2)^{-1/2}(\sigma^{\chi\varrho}-\sigma^{\varrho\chi});\quad \sigma^{\varrho\chi}=(2)^{-1/2}(\alpha-\sigma). \tag{6.7}$$

According to [11] a unique and selfconsistent solution of the corresponding equations can only be obtained by putting $s\equiv\sigma\equiv 0$. With these conditions and the abbreviated notation

$$\begin{aligned}&\varphi_1:=\varphi^{\varrho\varrho};\quad \sigma_1:=\sigma^{\varrho\varrho};\\ &\varphi_2:=a;\qquad \sigma_2:=\alpha;\\ &\varphi_3:=\varphi^{\chi\chi};\quad \sigma_3:=\sigma^{\chi\chi};\end{aligned} \tag{6.8}$$

the norm expression (6.4) goes over into

$$\langle a\,|\,a\rangle=\sum_{i=1}\sigma_i\,\varphi_i\,. \tag{6.9}$$

By a more detailed investigation of the corresponding equations in [11] it can further be demonstrated that in the case of very large elementary fermion masses very simple approximate solutions can be derived. In particular with

$$m:=m_1+m_2;\quad \Delta m:=m_2-m_1;\quad \eta:=m/\Delta m \tag{6.10}$$

we obtain in this approximation

$$\begin{aligned}&\varphi_2=-\eta\,\varphi_1;\quad \sigma_2=-\tfrac{1}{2}\eta\,\sigma_3;\\ &\varphi_3=\tfrac{1}{2}\eta^2\,\varphi_1;\quad \sigma_1=\tfrac{1}{2}\eta^2\,\sigma_3\end{aligned} \tag{6.11}$$

and

$$\sigma_3=\varphi_1^{+}. \tag{6.12}$$

Substitution of (6.11) and (6.12) in (6.9) yields

$$\langle a\,|\,a\rangle=\tfrac{3}{2}\eta^2\,\sigma_3\,\varphi_1=\tfrac{3}{2}\eta^2\,\varphi_1^{+}\,\varphi_1 \tag{6.13}$$

i.e. $|a\rangle$ can be normalized to unity as $\varphi_1^{+}\,\varphi_1$ and $\eta^2>0$.

We thus assume $\varphi_1=\mathcal{N}_\varphi\,\tilde{\varphi}_1$ where $\tilde{\varphi}_1$ is normalized to unity and obtain $\mathcal{N}_\varphi=(\frac{3}{2})^{-1/2}\,\eta^{-1}$ and $\langle a\,|\,a\rangle=1$.

All these relations can be substituted into (6.3) and (6.5) so that we finally obtain

$$\varphi^{rr'}=U^{rr'}\,\tilde{\varphi}_1;\quad \sigma^{rr'}=S^{rr'}\,\tilde{\varphi}_1^{+} \tag{6.14}$$

with

$$\begin{pmatrix}U^{11}\\ U^{21}\\ U^{12}\\ U^{22}\end{pmatrix}=\frac{1}{2}\left(\frac{3}{2}\right)^{-1/2}\begin{pmatrix}\eta^{-1}-\sqrt{2}+\frac{1}{2}\eta\\ \eta^{-1}\qquad -\frac{1}{2}\eta\\ \eta^{-1}\qquad -\frac{1}{2}\eta\\ \eta^{-1}+\sqrt{2}+\frac{1}{2}\eta\end{pmatrix} \tag{6.15}$$

and

$$\begin{pmatrix}S^{11}\\ S^{21}\\ S^{12}\\ S^{22}\end{pmatrix}=\frac{1}{4}\left(\frac{3}{2}\right)^{-1/2}\begin{pmatrix}\eta-\sqrt{2}+2\,\eta^{-1}\\ \eta\qquad -2\,\eta^{-1}\\ \eta\qquad -2\,\eta^{-1}\\ \eta+\sqrt{2}+2\,\eta^{-1}\end{pmatrix}, \tag{6.16}$$

where the difference between U and S comes from the difference between right-hand and left-hand solutions.

After having done these transformations and approximations, only the wave functions $\varphi_1\equiv\varphi_{1,\nu}^{(\pm)}(\boldsymbol{r},\boldsymbol{r}'\,|\,\boldsymbol{k})$ need to be determined. The corresponding equation is given in [11] by formula (3.21). A detailed investigation of of equation (3.21) is in preparation. Up to now only a corresponding scalar equation was studied in [11]. The results are: $\chi(0)$ must be finite and $\chi(\boldsymbol{r})$ is approximately given by $\exp\{-m\,|\boldsymbol{r}|\}$ for bound states.

Due to our threshold condition we are interested in bound state solutions of [11] (3.21) only, as all scattering state solutions contain elementary fermions and drop out below the threshold of elementary fermion production. We thus assume in accordance with Assumption 1:

Assumption 2: There are two bound state solutions of positive and negative class, respectively, which belong to positive and negative energies of the same scalar boson. Their wave functions are given by

$$\varphi_{1,0}^{(\pm)}(\boldsymbol{r},\boldsymbol{r}'\,|\,\boldsymbol{k})=e^{(\pm)i\boldsymbol{k}(\boldsymbol{r}+\boldsymbol{r}')/2}\,\chi_{\alpha\beta}\,\chi_0(\boldsymbol{r}-\boldsymbol{r}') \tag{6.17}$$

with

$$\chi_0(\boldsymbol{r})=(2m)^{3/2}\,\pi^{-3/4}\,e^{-\frac{1}{2}m^2\boldsymbol{r}^2} \tag{6.18}$$

and $\chi_{\alpha\beta}$ an antisymmetric spin tensor.

In this rough approximation the choice of a Gaussian internal wave function (6.18) instead of a hydrogen-like wave function is dictated only by the intention to simplify the subsequent calculations as much as possible. Similar results can be obtained with hydrogen-like wave functions. Furthermore, it is assumed that the internal wave function does not depend on $\boldsymbol{k}$. This, of course, is also an approximation which can be justified only in a low energy range. A further study of [11] (3.21) shows that our results are not qualitatively changed by the improvement of these wave functions.

Finally, we shortly discuss R^n_{uv}. For the determination of R^n_{uv} in principle Eqs. (3.10) must be used. The direct construction of such a dual system which satisfies (3.10) can be performed by Schmidt's orthogonalization procedure. In our case the application of this procedure is considerably simplified if the threshold condition is taken into account.

We decompose the set $\{C^{uv}_n\}$ into the subsets of bound boson states and of elementary fermion scattering states

$$\{C^{uv}_n\} = \{\varphi^{(\pm)}_0(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'} \mid \boldsymbol{k})\} \cup \{\varphi^{(\pm)}_\nu(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'} \mid \boldsymbol{k}), \nu \neq 0\}, \tag{6.19}$$

where on account of the preceding discussion we have

$$\{\varphi^{(\pm)}_0(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'} \mid \boldsymbol{k})\} = \{e^{(\pm)\, i\, \boldsymbol{k}(\boldsymbol{r}+\boldsymbol{r}')/2}\, U^{rr'}\, \chi_0(\underset{\alpha}{\boldsymbol{r}} - \underset{\beta}{\boldsymbol{r}'}),\ \forall \boldsymbol{k} \in R^3\}. \tag{6.20}$$

As was shown in the preceding section, for the transformation of the basic functional equation (1.15) into a cluster representation the full set (6.19) is needed. This, however, is only a formal prescription. The set of states $\{C^{uv}_n\}$ is always connected with the corresponding boson cluster operators b_n or $\delta/\delta b_n$, respectively, where the corresponding energies $E_b(n) \equiv E_b[(\pm), \nu, \boldsymbol{k}]$ are for $\nu \neq 0$ larger than the threshold energy $E_{\text{thr}} \equiv m$. Thus we may expect that for any total energy smaller than this threshold energy the operators b_n and $\delta/\delta b_n$ for all $n = [(\pm), \nu, k]$ with $\nu \neq 0$ can be suppressed from the beginning. This will be justified in Section 8. Taking this for granted we need only to work with the subset (6.20).

The same considerations can be performed for the dual set $\{R^n_{uv}\}$. We perform the decomposition

$$\{R^n_{uv}\} = \{\varrho^{(\pm)}_0(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'} \mid \boldsymbol{k})\} \cup \{\varrho^{(\pm)}_\nu(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'} \mid \boldsymbol{k}), \nu \neq 0\}, \tag{6.21}$$

where the first subset is the dual subset of the set (6.20) etc. The set is also fully needed for the transformation of Eq. (1.15) into a cluster representation. Then, independent of the special form of the wave functions of the dual set (6.21), the same arguments hold for the dual set as where given for the original set. Therefore all states (and corresponding operators) with $n = [(\pm), \nu, \boldsymbol{k}]$, $\nu \neq 0$ can be suppressed from the beginning and we really need only to consider the bound state subset of (6.21).

The omission of the subset for $\nu \neq 0$ by physical reasons is consistent with Schmidt's orthogonalization procedure. According to this procedure it is possible to first construct any arbitrary subspace of the total dual space and afterwards to complete the orthogonalization in the complement of this subspace. Thus we need only to explicitly construct the dual space for the set (6.20) which is by definition $\{\varrho^{(\pm)}_0(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'} \mid \boldsymbol{k})\}$ while the remaining states can be ignored although they are formally used in the expansion. The explicit construction of the dual subspace for the subset (6.20) can be easily performed if Assumption 2 is observed. According to this assumption the center of mass motion and the internal motion of the cluster can be separated in the original set (6.20). In consequence of this it is possible to also treat both these parts separately in the dual set. As far as the center of mass motion is concerned we shall apply the formalism of Hagedorn [89] for the construction of a dual set. With respect to the internal motion we proceed in analogy to the norm expression. We separate the auxiliary fields from the internal motion, so that finally only the dual state for $\chi_0(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'})$ is needed. Without any loss of generality this dual state can be assumed to be $\chi_0(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'})^+$. We thus obtain for the dual bound state subset

$$\{\varrho^{(\pm)}_0(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'} \mid \boldsymbol{k})\} \equiv \{(\pm)\, \omega(\boldsymbol{k})\, e^{(\pm)\, i\, \boldsymbol{k}(\boldsymbol{r}+\boldsymbol{r}')/2}\, S^{rr'}\, \chi_0(\underset{\alpha}{\boldsymbol{r}} - \underset{\beta}{\boldsymbol{r}'})^+\}, \tag{6.22}$$

where the factor $\pm\omega(\boldsymbol{k})$ in front of these wave functions arises from the construction of a complete set of orthonormalized plane waves for pointlike bosons of positive and negative class, while the factor $S^{rr'}$ is taken in analogy to the physical scalar products. All other functions of the complete dual

set which in particular are required for the fulfillment of the completeness relation are contained in the complementary set and need no explicit construction. Applying Hagedorn's formulae it can easily be proved that (6.22) is really the dual subspace of (6.20).

7. Three-Fermion Cluster Wave Function

The derivation of the three-fermion wave functions runs along the same lines as in the two-fermion case. Hence we can give an abridged version referring to Section 6.

Introducing the Jacobi coordinates

$$\boldsymbol{R} = \tfrac{1}{3}(\boldsymbol{r} + \boldsymbol{r}' + \boldsymbol{r}'');$$
$$\boldsymbol{x} = (\boldsymbol{r}' - \boldsymbol{r}''); \quad \boldsymbol{u} = \boldsymbol{r} - \tfrac{1}{2}(\boldsymbol{r}' - \boldsymbol{r}'') \tag{7.1}$$

we can write the three-fermion wave functions as follows

$$C_q^{rst} \equiv \varphi_\nu(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'}, \underset{\gamma}{\boldsymbol{r}''} \,|\, \boldsymbol{q}) = e^{i\boldsymbol{q}(\boldsymbol{r}+\boldsymbol{r}'+\boldsymbol{r}'')1/3}\, \chi_\nu(\underset{\alpha}{\boldsymbol{x}}, \underset{\beta}{\boldsymbol{u}} \,|\, \boldsymbol{q}) \tag{7.2}$$

where r, r', r'' are the auxiliary field indices, α, β, γ the spinor indices and ν the internal cluster state quantum number. The corresponding equations which can be obtained from (3.1) by projection were derived by Hailer [90] and will not explicitly be given here.

The source transformation (6.2) induces the transformation

(7.3)

$$\begin{pmatrix} \varphi^{111} \\ \varphi^{121} \\ \varphi^{112} \\ \varphi^{122} \\ \varphi^{211} \\ \varphi^{221} \\ \varphi^{212} \\ \varphi^{222} \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{\varphi}^{\varrho\varrho\varrho} \\ \hat{\varphi}^{\varrho\chi\varrho} \\ \hat{\varphi}^{\varrho\varrho\chi} \\ \hat{\varphi}^{\varrho\chi\chi} \\ \hat{\varphi}^{\chi\varrho\varrho} \\ \hat{\varphi}^{\chi\chi\varrho} \\ \hat{\varphi}^{\chi\varrho\chi} \\ \hat{\varphi}^{\chi\chi\chi} \end{pmatrix}$$

for the various auxiliary field components of (7.2). As in Sect. 6 only these indices are noted explicitly.

The norm of a three-fermion cluster state $|f\rangle$ is analogous to (6.4) given by

$$\langle f | f \rangle = \sum_{ijk} \sigma^{ikj}\, \varphi^{ikj} \tag{7.4}$$

and the left-hand solution components $\{\sigma^{ikj}\}$ transform with the same transformation law (7.3) as the right-hand solution components $\{\varphi^{ikj}\}$ do.

As in the two-fermion case also in the three-fermion case the unique solution of the eigenvalue equation requires the introduction of symmetry combinations. In these combinations the functions

$$z_1 = \hat{\varphi}^{\varrho\chi\varrho}; \quad z_1' = \hat{\varphi}^{\varrho\chi\chi};$$
$$z_2 = \hat{\varphi}^{\varrho\varrho\chi}; \quad z_2' = \hat{\varphi}^{\chi\varrho\chi};$$
$$z_3 = \hat{\varphi}^{\chi\varrho\varrho}; \quad z_3' = \hat{\varphi}^{\chi\chi\varrho} \tag{7.5}$$

participate. Hailer [90] has given two combinations

$$a_1 = \tfrac{1}{3}(z_1 + z_2 + z_3), \quad a_2 = \tfrac{1}{3}(2z_1 - z_2 - z_3), \tag{7.6}$$

which are symmetry adapted, and the same holds for the $\{z_i'\}$. These relations can be supplied by a third relation which need not be given here explicitly and which provide an orthogonal transformation

$$a_i = O_{ik}\, z_k; \quad a_i' = O_{ik}\, z_k' \tag{7.7}$$

so that

$$z_i = O_{ik}\, a_k; \quad z_i' = O_{ik}\, a_k' \tag{7.8}$$

holds. Only the total antisymmetric expressions a_1 and a_1' are nonvanishing, while $a_2 = a_2' = a_3 = a_3' = 0$ is required for a unique solution. This means

$$z_i = O_{i1}\, a_1; \quad z_i' = O_{i1}\, a_1'. \tag{7.9}$$

The same considerations hold for the components of the left-hand solutions, where we denote the symmetry adapted combinations by α_1 and α_1'. With the abbreviations

$$\varphi_1 := \hat{\varphi}^{\varrho\varrho\varrho}; \quad \sigma_1 := \sigma^{\varrho\varrho\varrho};$$
$$\varphi_2 := a_1; \quad \sigma_2 := \alpha_1;$$
$$\varphi_3 := a_1'; \quad \sigma_3 := \alpha_1';$$
$$\varphi_4 := \hat{\varphi}^{\chi\chi\chi}; \quad \sigma_4 := \sigma^{\chi\chi\chi} \tag{7.10}$$

the norm expression (7.4) goes over into

$$\langle f | f \rangle = \sum_{i=1}^{4} \sigma_i\, \varphi_i \tag{7.11}$$

and the corresponding equations are now equations for the determination of $\{\varphi_i\}$ or $\{\sigma_i\}$ respectively.

As in the two-fermion case also in the three-fermion case an approximation for large elementary fermion masses is possible. Without going into detail we give the result of this approximation

$$\varphi_2 = -\eta\, \varphi_1; \qquad \sigma_3 = -\eta\, \sigma_4;$$
$$\varphi_3 = \tfrac{3}{2}\eta^2\, \varphi_1; \qquad \sigma_2 = \tfrac{3}{2}\eta^2\, \sigma_4;$$
$$\varphi_4 = -\tfrac{9}{2}\eta^3\, \varphi_1; \qquad \sigma_1 = -\tfrac{9}{2}\eta^3\, \sigma_4 \tag{7.13}$$

and

$$\sigma_4 = \varphi_1^+ . \tag{7.13}$$

Substitution of (7.12) and (7.13) into (7.11) yields for $\eta < 0$

$$\begin{aligned} \langle f | f \rangle &= -12\eta^3 \sigma_4 \varphi_1 = -12\eta^3 \varphi_1^+ \varphi_1 \\ &= 12 |\eta^3| \, |\mathcal{N}_\varphi|^2 \tilde{\varphi}_1^+ \tilde{\varphi}_1 = 1 , \end{aligned} \tag{7.14}$$

where $\tilde{\varphi}_1$ is normalized to unity. Thus $\mathcal{N}_\varphi$ must take the value $\mathcal{N}_\varphi = 12^{-1/2} |\eta|^{-3/2}$.

All these relations can be substituted into (7.3) and the corresponding equation for the $\{\sigma^{ijk}\}$ and yield

$$\varphi^{rr'r''} = U^{rr'r''} \tilde{\varphi}_1 ; \quad \sigma^{rr'r''} = S^{rr'r''} \tilde{\varphi}_1^+ \tag{7.15}$$

with the abbreviation $\eta = |\eta|$:

$$\begin{pmatrix} U^{111} \\ U^{121} \\ U^{112} \\ U^{122} \\ U^{211} \\ U^{221} \\ U^{212} \\ U^{222} \end{pmatrix} = \frac{1}{4\sqrt{6}} \begin{pmatrix} \eta^{-3/2} & -\eta^{-1/2} & +\frac{3}{2}\eta^{1/2} & -\frac{9}{2}\eta^{3/2} \\ \eta^{-3/2} & -\frac{1}{3}\eta^{-1/2} & -\frac{1}{2}\eta^{1/2} & +\frac{9}{2}\eta^{3/2} \\ \eta^{-3/2} & -\frac{1}{3}\eta^{-1/2} & -\frac{1}{2}\eta^{1/2} & +\frac{9}{2}\eta^{3/2} \\ \eta^{-3/2} & +\frac{1}{3}\eta^{-1/2} & -\frac{1}{2}\eta^{1/2} & -\frac{9}{2}\eta^{3/2} \\ \eta^{-3/2} & -\frac{1}{3}\eta^{-1/2} & -\frac{1}{2}\eta^{1/2} & +\frac{9}{2}\eta^{3/2} \\ \eta^{-3/2} & +\frac{1}{3}\eta^{-1/2} & -\frac{1}{2}\eta^{1/2} & -\frac{9}{2}\eta^{3/2} \\ \eta^{-3/2} & +\frac{1}{3}\eta^{-1/2} & -\frac{1}{2}\eta^{1/2} & -\frac{9}{2}\eta^{3/2} \\ \eta^{-3/2} & +\eta^{-1/2} & +\frac{3}{2}\eta^{1/2} & +\frac{9}{2}\eta^{3/2} \end{pmatrix} \tag{7.16}$$

and

$$\begin{pmatrix} S^{111} \\ S^{121} \\ S^{112} \\ S^{122} \\ S^{211} \\ S^{221} \\ S^{212} \\ S^{222} \end{pmatrix} = \frac{1}{4\sqrt{6}} \begin{pmatrix} -\frac{9}{2}\eta^{3/2} & +\frac{3}{2}\eta^{1/2} & -\eta^{-1/2} & +\eta^{-3/2} \\ -\frac{9}{2}\eta^{3/2} & +\frac{1}{2}\eta^{1/2} & +\frac{1}{3}\eta^{-1/2} & -\eta^{-3/2} \\ -\frac{9}{2}\eta^{3/2} & +\frac{1}{2}\eta^{1/2} & +\frac{1}{3}\eta^{-1/2} & -\eta^{-3/2} \\ -\frac{9}{2}\eta^{3/2} & -\frac{1}{2}\eta^{1/2} & +\frac{1}{3}\eta^{-1/2} & +\eta^{-3/2} \\ -\frac{9}{2}\eta^{3/2} & +\frac{1}{2}\eta^{1/2} & +\frac{1}{3}\eta^{-1/2} & -\eta^{-3/2} \\ -\frac{9}{2}\eta^{3/2} & -\frac{1}{2}\eta^{1/2} & +\frac{1}{3}\eta^{-1/2} & +\eta^{-3/2} \\ -\frac{9}{2}\eta^{3/2} & -\frac{1}{2}\eta^{1/2} & +\frac{1}{3}\eta^{-1/2} & +\eta^{-3/2} \\ -\frac{9}{2}\eta^{3/2} & -\frac{3}{2}\eta^{1/2} & -\eta^{-1/2} & -\eta^{-3/2} \end{pmatrix} \tag{7.17}$$

After having done these transformations and approximations, only the wave functions $\varphi_1 \equiv \varphi_{1,v}(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'}, \underset{\gamma}{\boldsymbol{r}''} \,|\, \boldsymbol{q})$ need to be determined. According to Assumption 1 for $v = 0$ we have a bound state of elementary fermions, while for $v \neq 0$ all states are scattering states of these elementary fermions. Detailed investigations of the corresponding wave equation are in preparation. At present, however, we must guess the results. By analogous reasoning to the two-fermion case we here make the following assumption:

Assumption 3: The three-fermion bound state wave functions are given by

$$\varphi_{1,0}(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'}, \underset{\gamma}{\boldsymbol{r}''} \,|\, t\, \boldsymbol{q}) = e^{i\boldsymbol{R}\boldsymbol{q}} \chi^t_{\alpha\beta\gamma} \chi_0(\boldsymbol{x}, \boldsymbol{u}) \tag{7.18}$$

with $t \equiv$ spin direction index and with

$$\chi_0(\boldsymbol{x}, \boldsymbol{u}) := (2m)^3 \pi^{-3/2} e^{-\frac{1}{2}m^2\boldsymbol{x}^2} e^{-\frac{1}{2}m^2\boldsymbol{u}^2}, \tag{7.19}$$

where $\chi^t_{\alpha\beta\gamma}$ transforms isomorphic to a spin 1/2 spinor wave function $\psi^t_\alpha(\boldsymbol{q})$.

For the antifermion cluster function $\{\bar{C}^{abc}_w\}$ the same considerations can be applied and therefore need not be given explicitly.

As far as the construction of the dual state systems $\{R^q_{rst}\}$ or $\{\bar{R}^w_{abc}\}$ is concerned, all arguments given for the two-fermion case can literally be taken over if the term "bound boson states" is replaced by "bound three fermion states" etc. Thus these arguments also need not be repeated here. We give only the result. The physical relevant subset of dual three fermion bound states reads

$$\begin{aligned} &\{\varrho_0(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'}, \underset{\gamma}{\boldsymbol{r}''} \,|\, t\, \boldsymbol{q})\} \\ &\quad := \{e^{-i\boldsymbol{R}\boldsymbol{q}} S^{rr'r''} \bar{\chi}_0(\underset{\alpha\beta\gamma}{\boldsymbol{x}, \boldsymbol{u}} \,|\, t\, \boldsymbol{q}), \forall \boldsymbol{q} \in R^3, t\} \end{aligned} \tag{7.20}$$

while the original set is defined by

$$\begin{aligned} &\{\varphi_0(\underset{\alpha}{\boldsymbol{r}}, \underset{\beta}{\boldsymbol{r}'}, \underset{\gamma}{\boldsymbol{r}''} \,|\, t\, \boldsymbol{q})\} \\ &\quad := \{e^{i\boldsymbol{R}\boldsymbol{q}} U^{rr'r''} \chi_0(\underset{\alpha\beta\gamma}{\boldsymbol{x}, \boldsymbol{u}} \,|\, t\, \boldsymbol{q}), \forall \boldsymbol{q} \in R^3, t\}, \end{aligned} \tag{7.21}$$

where the definitions (7.16), (7.17) etc. have to be observed.